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Similarity solutions to three-dimensional nonlinear diffusion equations

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Abstract. The similarity transformation method has been used to solve three-dimensional nonlinear diffusion equations. The general Lie group is calculated. Exact solutions to the cylindrical and spherical symmetry cases are found, and their relations to some real physical processes are discussed.

Nonlinear diffusion equations are widely investigated and have applications in numerous fields [1-3]. Most of these studies are concerned with 1D or quasi-1D (e.g. *n*D spherical symmetry) equations. As to higher-dimensional equations, people generally consider problems such as existence of solutions, conditions for finite propagation speed [4], existence of travelling wave solutions [5], and asymptotic behaviour as $t \to \infty$ [6], rather than exact solutions. The similarity transformation (ST) method has many applications when dealing with differential equations and related physical problems [3, 7, 8]. Especially in the nonlinear case, it can sometimes help us in finding physically meaningful exact solutions. It is worth noting that, in some circumstances, solutions of nonlinear equations with appropriate boundary conditions converge to ST forms [9] as $t \to \infty$.

The equation we are going to consider is

$$K(u)\partial_t u = \mathrm{div}[\mathrm{grad}(u)] \tag{1}$$

or the equivalent equation

$$\partial_t v = \mathrm{div}[D(v)\,\mathrm{grad}(v)] \tag{1'}$$

with $v = M(u) = \int K(u)\,\mathrm{d}u$, $[D(v)]^{-1} = M'[M^{-1}(v)]$. We consider the following infinitesimal transformation:

$$\begin{aligned}
x_1 &= x + \varepsilon X(x, y, z, t, u) + \mathrm{O}(\varepsilon^2)\\
y_1 &= y + \varepsilon Y(x, y, z, t, u) + \mathrm{O}(\varepsilon^2)\\
z_1 &= z + \varepsilon Z(x, y, z, t, u) + \mathrm{O}(\varepsilon^2)\\
t_1 &= t + \varepsilon T(x, y, z, t, u) + \mathrm{O}(\varepsilon^2)\\
u_1 &= u + \varepsilon U(x, y, z, t, u) + \mathrm{O}(\varepsilon^2)
\end{aligned} \tag{2}$$

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where X (also denoted as $[x]$) is called the infinitesimal element of x, and so on. We assume that (2) leaves (1) invariant, i.e. (1) holds when x and other variables are replaced by x_1 and others. This invariance gives

$$K'(u)U\partial_t u + K(u)[\partial_t u] = [\partial_{xx}u] + [\partial_{yy}u] + [\partial_{zz}u] \quad (3)$$

(see, for example, [3, p 155]). Suppose that the derivatives of u are independent, then we obtain from (3) a system of equations (the determining equations) about X, Y, etc. The resolution of this system gives (i) $K(u)$ arbitrary, (ii) $K(u) = K_0 \exp(au)$, and (iii) $k(u) = k_0(u_0 + u)^m$, k_0, a, u_0 and m are arbitrary constants. As it is difficult to get an interesting result from cases (i) and (ii), we restrict ourselves to case (iii) hereafter, and we have

$$\begin{aligned} X &= a_4 x + a_3 y + a_2 z + c_1 \\ Y &= -a_3 x + a_4 y + a_1 z + c_2 \\ Z &= -a_2 x - a_1 y + a_4 z + c_3 \\ T &= (mb + 2a_4)t + c_4 \\ U &= b(u_0 + u) \end{aligned} \quad (4)$$

where a_α, c_α, $a = 1, 2, 3, 4$, and b are nine arbitrary parameters. Equation (1) is hence invariant under the following infinitesimal generators (also called vector fields):

$$\begin{aligned} &C_\alpha = \partial_{x\alpha} \qquad \alpha = 1, 2, 3, 4 \qquad x_4 \equiv t \qquad \boldsymbol{A} = \boldsymbol{X} \times \boldsymbol{\nabla} \\ &A_4 = \boldsymbol{x} \cdot \boldsymbol{\nabla} + 2t\partial_t \qquad B = t\partial_t + u/m\partial_u. \end{aligned} \quad (5)$$

A straightforward calculation gives the following commutating relations which define the infinitesimal Lie group:

$$\begin{aligned} &[C_\alpha, C_\beta] = 0 && [C_i, A_4] = C_i && \\ &[C_4, A_i] = 0 && [C_4, A_4] = 2C_4 && [C_i, A_j] = -\varepsilon_{ijk}C_k \\ &[C_i, B] = [A_\alpha, B] = 0 && [C_4, B] = C_4 && \\ &[A_i, A_j] = \varepsilon_{ijk}A_k && [A_4, A_i] = A_i && \alpha, \beta = 1, \ldots, 4 \qquad i, j, k = 1, 2, 3. \end{aligned} \quad (6)$$

We see from (4) or (5) that equation (1) with $K(u)$ given above possesses very abundant symmetries. To obtain final solutions we restrict our consideration to some special cases. At first, consider the cylindrical one. Without loss of generality, let $k_0 = 1$, $u_0 = 0$, $\alpha_3 \neq 0$, and the other eight parameters are set to zero in (4). This corresponds to choosing only the generator A_3 in (5). Through the characteristic equations

$$\mathrm{d}x/X = \mathrm{d}y/Y = \mathrm{d}z/Z = \mathrm{d}t/T = \mathrm{d}u/U \quad (7)$$

we obtain the cylindrical symmetry equation:

$$u^m \partial_t u = (1/x)\partial_x u + \partial_{xx}u + \partial_{yy}u \qquad x > 0. \quad (8)$$

Let $(\eta_1, \eta_2, \eta_3, V)$ be the infinitesimal elements of (x, y, t, u). By a similar method, we find the following results:

(*a*) m is arbitrary:

$$\begin{aligned} \eta_1 &= ax \\ \eta_2 &= ay + c_1 \\ \eta_3 &= (md + 2a)t + c_2 \\ V &= du \end{aligned} \quad (9)$$

(*b*) $m=4$:

$$\begin{aligned} \eta_1 &= (a-2by)x \\ \eta_2 &= b(x^2-y^2)+ay+c_1 \\ \eta_3 &= (4d+2a)t+c_2 \\ V &= (d+by)u \end{aligned} \qquad (10)$$

where a, b, c_1, c_2, and d are five aribtrary parameters. We consider only (*b*), and let $c_1=a=d=0$, $c_2=1$, $b\neq 0$. Through this calculation we obtain $u=f(s,\omega)/\sqrt{x}$, where s, ω are called similarity variables and given by

$$s=b(x^2+y^2)/x \qquad \omega=[y-bt(x^2+y^2)]/x$$

and f (the similarity form) satisfies

$$sf_\omega f^4+\tfrac{1}{4}f+s^2f_{ss}+(1+\omega^2)f_{\omega\omega}+2s\omega f_{s\omega}+2sf_s+2\omega f_\omega=0$$

which, under another ST, gives $f(s,\omega)=s^{-1/4}g(\omega)$, where g is governed by the ordinary differential equation:

$$(1+\omega^2)g''+(g^4+\tfrac{3}{2}\omega)g'+\tfrac{1}{16}g=0.$$

We find that this equation admits

$$g(\omega)=(\tfrac{4}{5}\omega)^{-1/4}$$

as a special solution. Thus we have the following exact solution to equation (8):

$$u(x,y,t)=\{(4b/5)(x^2+y^2)[y-bt(x+y)]\}^{-1/4}. \qquad (11)$$

In fact, for any value of m, (8) admits a solution as

$$u(x,y,t)=[(1+m)/mb](x^2+y^2)^{(2-m)/m}[y-b(t-t_0)(x^2+y^2)]^{-1/m} \qquad (12)$$

where b is an arbitrary constant and t_0 corresponds to time-translational invariance. The corresponding solution to (1′)

$$\partial_t v=(1/x)\partial_x(v^n\partial_x v)+\partial_y(v^n\partial_y v) \qquad D(v)=v^n$$

is given by

$$v(x,y,t)=(nc)^{1/n}(x^2+y^2)^{-(3n+2)/2n(n+1)}[y+c(t-t_0)(x^2+y^2)]^{1/n} \qquad (12')$$

with $v=u^{m+1}$, $n=-m/(m+1)$, $c=-b$. We now discuss briefly the behaviours of (12′) for some particular value of n, c.

(*a*) $n=2$ ($m=-\frac{2}{3}$), $c>0<t_0$. In this case, for $t<t_0$, the domain of definition is expressed by the sphere $y+c(t-t_0)(x^2+y^2)\geq 0$ whose radius is $1/[2c(t_0-t)]$, and which expanded at a speed $[2c(t_0-t)]^{-2}$, and $v=0$ outside the domain. With the singularity at point (0, 0), we say physically that v describes a source distribution, having a delta point at (0, 0), which diffuses with a finite propagation speed $[\sim(t_0-t)^{-2}]$ for $t<t_0$.

(*b*) $n=1$ ($m=-0.5$), $c<0<t_0$. In this case, $D(v)=-v<0$ and thus v describes the field of an attractive source, i.e. an abyss whose centre is at the original point. The domain of the definition of v is confined to within the sphere

$$y-b(t-t_0)(x^2+y^2)\geq 0$$

which constricts when t increases, and tends to the point abyss at the origin as $t\rightarrow\infty$.

Now we consider the spherical case and generalise to the *n*D space:

$$u^m \partial_t u = \frac{N-1}{r} \partial_r u + \partial_{rr} u, \qquad r > 0.$$

The ST analysis gives $u(r, t) = f(\theta) t^{-p}$, $\theta = r t^{-g}$, and if

(*a*) $p = N/[(2-N)m+2]$, $q = (m+1)/[(2-N)m+2]$, then

$$f' + \frac{p}{N} \theta f^{m+1} + c_1 = 0.$$

If $c_1 = 0$, we obtain the solution describing delta source diffusion with moving boundary condition [4]

$$u(r, t) = (t - t_0)^{-p/(n+1)} \left(a^2 - \frac{cr^2}{(t-t_0)^{2g}} \right)^{1/n}$$

where $c = pn/[2N(n+1)]$, t_0 and a are two constants.

(*b*) If $p = 1$, $q = (m+1)/2$, then

$$\theta f' + (N-2) f + \tfrac{1}{2} \theta^2 f^{m+1} + c_2 = 0. \tag{13}$$

If $c_2 = 0$, (13) is a Bernoulli equation. The general solution is

$$u(r, t) = t^{-1/(n+1)} \left(B^{(2-N)n/(n+1)} - \frac{n\theta^2}{2(2+nN)} \right)^{1/n}. \tag{14}$$

This expression generalises the result of [10], where $B = 0$, and was used to describe the so-called 'waiting time' effect. Abundant physical processes have their correspondence to the appropriate values of (n, N, B).

If $c_2 \neq 0$, and if $N = 3$, $m = 1$ ($n = 0.5$), then (13) is exactly solvable and we find

$$\left.\begin{aligned} u(r, t) &= \sqrt{c_2}/r \tanh[\sqrt{c_2}\, r/2(t-t_0) - B] \\ v(r, t) &= c_2/r^2 \tanh^2[\sqrt{c_2}\, r/2(t-t_0) - B] \end{aligned}\right\} \quad c_2 > 0. \tag{15}$$

This is a front wave solution, but with an attenuation factor $1/r$ as it is in 3D space. A simple transformation from (15) gives the following soliton-like solution:

$$g(r, t) = c/2\, e^{-2t} \operatorname{sech}^2(\sqrt{c}\, r\, e^{-t} - B) \tag{16}$$

to the nonlinear reaction-diffusion equation

$$\partial_t g = \partial_r (g^{-1} \partial_r g) + 2g - r \partial_r g. \tag{17}$$

A real soliton does not change its form with time. Equation (16) differs from a real soliton in the fact that it has the attenuation factor $\exp(-2t)$, and the enlargement of the half-width as time increases.

In summary, using the ST, we have obtained the infinitesimal Lie group (6) for the 3D NLDE (1). Applying the ST successively to the cylindrical symmetry equation (8) we have obtained an exact solution (12). The exact solutions (14) and (15) to the spherical symmetry case were also found by ST. We have discussed the possible relations between these solutions and some real physical processes. Finally, an interesting result was obtained: a soliton-like solution (16) to the reaction–diffusion equation (17). We note that we have only chosen a very particular case (A_3) among the abundant symmetries (5). We hope therefore that many other symmetries will provide us with some new interesting and exact results.

References

[1] Munier A, Burgan J R, Gutierrez J, Fijalkow E and Feix M R 1981 SIAM J. Appl. Math. **40** 191–207
[2] Dresner L 1983 *Similarity Solutions of NLPDEs* (Pitman Advanced Publishing Program) (New York: Pitman)
[3] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Berlin: Springer)
[4] Levine H A and Sacks E 1984 *J. Diff. Eq.* **52** 135–61
[5] Nimmo J J C and Crighton D G 1985 *Proc. R. Soc.* A **384** 381–401
[6] Holland C J and Berryman J G 1985 *J. Math. Phys.* **26** 660–3
[7] Burgan J R *et al* 1978 *Group Transformation for Phase Space Fluids in Strongly Coupled Plasma* (New York: Plenum) pp 597–641
[8] Munier A, Feix M, Fijiakov E, Burgan J R and Gutierrez J 1979 *Astron. Astrophys.* **78** 65–74
[9] Grundy R E and McLaughlin R 1982 *Proc. R. Soc.* A **381** 395–406
[10] Aronson D G 1970 *SIAM J. Appl. Math.* **19** 299–307